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Some properties of the Sobolev–Laguerre polynomials

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Abstract

In this paper, we study the case $\alpha = 0$ of the Sobolev–Laguerre polynomials. We determine a generating function for the polynomials and an expansion formula © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

The so-called Sobolev–Laguerre polynomials have been discussed by a number of different writers. The (general) Sobolev–Laguerre polynomials are orthogonal with respect to the inner product

$$(^*)\langle f,g\rangle = \int_0^\infty x^\alpha e^{-x} f(x)g(x)\,dx + \lambda \int_0^\infty x^\alpha e^{-x} f'(x)g'(x)\,dx, \quad \lambda \geqslant 0.$$

The polynomials were exhaustively studied by Brenner in the case $\alpha = 0$ who found a (not very useful) explicit formula for the polynomials [1]. For the more recently investigated case of general α, the article by Marcellán et al. [4] provides an excellent and timely bibliography, as well as a number of original findings.

In this paper, we study the case $\alpha = 0$. The notation we use for these polynomials is $L_n^\lambda(x)$. We determine an elegant generating function for the polynomials and some expansions involving the polynomials. These results seem to be new.

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2. Results

We will employ the notation: if **A** is a matrix with elements $a_{i,j}$, then A will denote the determinant whose elements are $a_{i,j}$, and vice versa.

In our determinants and matrices, i will denote row position, j column position.

We can write $L_n^\lambda(x)$ as an $(n+1)\times(n+1)$ Gram-type determinant of moments and powers of x,

$$L_n^\lambda(x) = \begin{vmatrix} & 1 \\ & x \\ (c_{i,j})_{\substack{i=0\dots n \\ j=0\dots n-1}} & x^2 \\ & \vdots \\ & x^n \end{vmatrix}, \tag{1}$$

where

$$c_{i,j} = \langle x^i, x^j\rangle = \begin{cases} (i+j)! & i=0 \text{ or } j=0, \\ (i+j)! + \lambda ij(i+j-2)! & \text{otherwise.} \end{cases} \tag{2}$$

(It is easily verified that (1) provides a legitimate expression for $L_n^\lambda(x)$ by taking the inner product of the determinant with x^i, $i=0,1,\dots,n-1$. The last column of the resulting determinant is equal to another column, so the result is 0.)

With these polynomials are associated two normalization constants, $k_n^{(\lambda)}$, the coefficient of the highest power of x in $L_n^{(\lambda)}$, and

$$h_n^{(\lambda)} = \langle L_n^{(\lambda)}, L_n^{(\lambda)}\rangle. \tag{3}$$

Taking inner products of the determinant (1) with $L_n^{(\lambda)}$ shows

$$h_n^{(\lambda)} = k_n^{(\lambda)} k_{n+1}^{(\lambda)}, \tag{4}$$

a useful formula.

It is convenient to factor $(i-1)!$ out of the ith row, $i=1,2,\dots,n+1$, and $(j-1)!$ out of the jth column, $j=1,2,\dots,n$ of $L_n^{(\lambda)}$ and to consider instead the polynomial

$$V_n = \begin{vmatrix} & 1 \\ & x \\ (d_{i,j})_{\substack{i=0\dots n \\ j=0\dots n-1}} & x^2/2 \\ & \vdots \\ & x^n/n! \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \lambda+2 & \lambda+3 & \dots & x \\ 1 & \lambda+3 & 2\lambda+6 & \dots & x^2/2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & . & . & . & x^n/n! \end{vmatrix} = \rho_n L_n^{(\lambda)}, \tag{5}$$

where

$$d_{i,j} = \begin{cases} 1 & i=0 \text{ or } j=0, \\ [(i+j)! + \lambda ij(i+j-2)!]/i!j! & \text{otherwise,} \end{cases} \tag{6}$$

and

$$\rho_n = \left(\prod_{j=1}^{n} j!(j-1)!\right)^{-1}. \tag{7}$$

We will investigate three $(n+1)\times(n+1)$ determinants, which are auxiliary polynomials in x of degree n:

$$P_n(x) = \begin{vmatrix} (d_{i,j})_{\substack{i=0\dots n\\ j=0\dots n-1}} & \begin{matrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{matrix} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \lambda+2 & \lambda+3 & \dots & x \\ 1 & \lambda+3 & 2\lambda+6 & \dots & x^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & . & . & . & x^n \end{vmatrix}, \tag{8}$$

$$Q_n(x) = \begin{vmatrix} (d_{i,j})_{\substack{i=1\dots n+1\\ j=1\dots n}} & \begin{matrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{matrix} \end{vmatrix} = \begin{vmatrix} \lambda+2 & \lambda+3 & \lambda+4 & \dots & 1 \\ \lambda+3 & 2\lambda+6 & 3\lambda+10 & \dots & x \\ \lambda+4 & 3\lambda+10 & 6\lambda+20 & \dots & x^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & . & x^n \end{vmatrix}, \tag{9}$$

$$R_n(x) = \begin{vmatrix} (d_{i,j})_{\substack{i=0\dots n\\ j=1\dots n}} & \begin{matrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{matrix} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda+2 & \lambda+3 & \lambda+4 & \dots & x \\ \lambda+3 & 2\lambda+6 & 3\lambda+10 & \dots & x^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & . & x^n \end{vmatrix}. \tag{10}$$

Note that $L_n^{\lambda}(x)$ is essentially the inverse Laplace transform of $(1/p)P_n(1/p)$. This crucial fact will be used later.

We will also investigate the three following $n\times n$ constant (independent of x) determinants:

$$A_n = \left|(d_{i,j})_{\substack{i=0\dots n-1\\ j=0\dots n-1}}\right| = \begin{vmatrix} 1 & 1 & 1 & \dots \\ 1 & \lambda+2 & \lambda+3 & \dots \\ 1 & \lambda+3 & 2\lambda+6 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}, \tag{11}$$

$$B_n = \left|(d_{i,j})_{\substack{i=1\dots n\\ j=1\dots n}}\right| = \begin{vmatrix} \lambda+2 & \lambda+3 & \lambda+4 & \dots \\ \lambda+3 & 2\lambda+6 & 3\lambda+10 & \dots \\ \lambda+4 & 3\lambda+10 & 6\lambda+20 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}, \tag{12}$$

$$C_n = \left|(d_{i,j})_{\substack{i=0\dots n-1\\ j=1\dots n}}\right| = \begin{vmatrix} 1 & 1 & 1 & \dots \\ \lambda+2 & \lambda+3 & \lambda+4 & \dots \\ \lambda+3 & 2\lambda+6 & 3\lambda+10 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}. \tag{13}$$

Our proofs of results pertaining to these quantities will utilize the three $n \times n$ elementary matrices

$$\boldsymbol{E}_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ & & & \ddots & & \\ & & & & 1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{bmatrix}, \qquad \boldsymbol{F}_n = \boldsymbol{E}_n^{\mathrm{T}},$$

$$\boldsymbol{G}_n = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \ddots & & \\ & & & & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}. \tag{14}$$

Note that $\boldsymbol{G}_n$ differs from $\boldsymbol{F}_n$ by a single element, having 0 in the $(n-1, n)$ position, rather than -1. Note $E_n = F_n = G_n = 1$.

Theorem 1. *Let*

$$\lambda = 4\sinh^2\theta. \tag{15}$$

Then

$$A_n = \frac{\cosh(2n-1)\theta}{\cosh\theta}, \tag{16}$$

$$B_n = \frac{\sinh(2n+2)\theta}{\sinh 2\theta}, \tag{17}$$

$$C_n = 1. \tag{18}$$

Proof. A simple computation shows that

$$\boldsymbol{E}_n \boldsymbol{C}_n \boldsymbol{F}_n = \begin{bmatrix} 1 & \lambda+1 & 1 & 1 & \dots \\ 0 & 1 & \lambda+2 & \lambda+3 & \dots \\ 0 & 1 & \lambda+3 & 2\lambda+6 & \dots \\ 0 & 1 & \lambda+4 & 3\lambda+10 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}^{\mathrm{T}}. \tag{19}$$

Taking determinants gives $C_n = C_{n-1}$ and induction immediately gives the third statement.

Next, we have

$$\boldsymbol{E}_n\boldsymbol{A}_n\boldsymbol{F}_n=\begin{bmatrix}1&0&0&0&\cdots\\0&\lambda+1&1&1&\cdots\\0&1&\lambda+2&\lambda+3&\cdots\\0&1&\lambda+3&2\lambda+6&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{bmatrix}. \tag{20}$$

Taking determinants, expanding the determinant on the right by minors of the first column, and then expanding that determinant by minors of its first column give

$$A_n=A_{n-1}+\lambda B_{n-2}. \tag{21}$$

Applying Sylvester's theorem (see [7, p. 251]) to A_n and utilizing the known value of C_n give

$$A_nB_{n-2}=A_{n-1}B_{n-1}-1. \tag{22}$$

Solving this equation for B_{n-2} and substituting it into the previous equation yields

$$A_n^2=A_{n+1}A_{n-1}-\lambda. \tag{23}$$

Induction on this equation gives the A_n result.

Finally, substituting the known value of A_n into (21) gives the B_n result.

Theorem 2. *Let*

$$D=x^2-\lambda(1-x).$$

Then

(i)

$$P_n=\frac{(-1)^n\lambda(x-1)}{D}+\frac{x(x-1)^n}{D}[(x-1)A_n+A_{n+1}],$$

(ii)

$$Q_n=\frac{(-1)^n}{D}+\frac{(x-1)^{n+1}}{\lambda D}[(\lambda+x)A_{n+2}-xA_{n+1}],$$

(iii)

$$R_n=(-1)^n\left[\frac{A_{n+1}-(1-x)A_{n+2}-x(1-x)^{n+1}}{D}\right].$$

Proof. Considering the determinants of the matrices $\boldsymbol{E}_{n+1}\boldsymbol{P}_n\boldsymbol{G}_{n+1}$ and $\boldsymbol{E}_{n+1}\boldsymbol{R}_n\boldsymbol{G}_{n+1}$ in much the same way as in the proof of Theorem 1 shows that

$$P_n=(x-1)P_{n-1}+\lambda x(x-1)Q_{n-2}, \tag{24}$$

$$R_n=(x-1)R_{n-1}+(-1)^nA_{n+1}. \tag{25}$$

Applying Sylvester's theorem to P_n gives

$$P_n B_{n-1} = A_n x Q_{n-1} - R_{n-1}. \tag{26}$$

We write (25) as

$$\frac{R_n}{(x-1)^n} - \frac{R_{n-1}}{(x-1)^{n-1}} = \frac{(-1)^n \cosh(2n+1)\theta}{\cosh\theta (x-1)^n}. \tag{27}$$

Expressing the hyperbolic functions as exponentials and iterating this expression, i.e., replacing n by k and summing from $k=1$ to n (note (25) requires interpreting $R_0 = 1$), give, after some numbing algebra, (iii).

We now solve (24) for Q_{n-2}, replace n by $n+1$, substitute the result into (26), substitute the just obtained value of R_n into (26), and finally solve for B_{n-1} from (21) and substitute that into the equation. We obtain the following first-order difference equation for P_n:

$$\frac{P_n A_{n+1}}{(x-1)^n} - \frac{P_{n+1} A_n}{(x-1)^{n+1}} = \frac{\lambda}{D}\left[(-1)^n \frac{A_n}{(x-1)^n} + (-1)^n \frac{A_{n+1}}{(x-1)^{n-1}}\right] - \frac{\lambda x}{D}. \tag{28}$$

It is easily verified, using the known properties of A_n, that particular solutions corresponding to the first and to the second term on the right are, respectively,

$$P_n^{(1)} = \frac{\lambda(-1)^n (x-1)}{D}, \qquad P_n^{(2)} = \frac{A_{n+1} x (x-1)^n}{D}. \tag{29}$$

A homogeneous solution is

$$P_n^{(3)} = A_n (x-1)^n. \tag{30}$$

We write

$$P_n = P_n^{(1)} + P_n^{(2)} + C P_n^{(3)}. \tag{31}$$

C is determined by setting $n=1$ and using $P_1 = x-1$. The result is (i). Finally, using the above value of P_n we solve for Q_n from (24). The result is then simplified by expressing A_{n+3} in terms of A_{n+2} and A_{n+1} and the result is (ii).

In what follows, we let

$$a = \frac{-\lambda - 2 + \sqrt{\lambda^2 + 4\lambda}}{2}, \qquad \rho_n = \left(\prod_{j=1}^{n} j!(j-1)!\right)^{-1}. \tag{32}$$

Theorem 3. *For* $|t| < \min\{1, |a|/|x-1|\}$,

$$h(x,t) = \sum_{n=0}^{\infty} P_n(x)(-t)^n, \tag{33}$$

where

$$h(x,t) = \frac{[(1-t)^2 + \lambda t(x-1) + tx(1-t)]}{(1-t)U}, \tag{34}$$

and

$$U = (t(x-1)+1)^2 + \lambda t(x-1). \tag{35}$$

Proof. Easy algebra gives

$$\begin{aligned} \sum_{n=0}^{\infty} A_n(-t)^n(x-1)^n &= \frac{1+t(x-1)(\lambda+1)}{U}, \\ \sum_{n=0}^{\infty} A_{n+1}(-t)^n(x-1)^n &= \frac{1+t(x-1)}{U}. \end{aligned} \tag{36}$$

(Note $A_0 = 1$.) Multiplying Theorem 2(i) by $(-t)^n$ and summing gives the result. (It requires a rather formidable amount of algebra to show the factor D cancels out of the right-hand side.) The region of convergence of the series is easily determined by examining the location of the poles of the left-hand side. These occur at $t=1$, $t=a/(x-1)$, $t=1/[a(x-1)]$. Since $|a|<1<|1/a|$, the region of convergence is as indicated.

Theorem 4. *For* $|t|<|a|$,

$$\frac{a e^{xt/(t+a)} - e^{xt/(t+1/a)}}{(a-1)(1-t)} = \sum_{n=0}^{\infty} \rho_n L_n^{(\lambda)}(x)(-t)^n. \tag{37}$$

Proof. We observe that

$$\frac{1}{p} P_n\left(\frac{1}{p}\right) = \rho_n \mathscr{L}\{L_n^{(\lambda)}(x)\}, \tag{38}$$

where $\mathscr{L}$ is the Laplace transform operator,

$$\mathscr{L}\{f(x)\} = \int_0^{\infty} e^{-px} f(x)\,dx. \tag{39}$$

Replacing x by $1/p$ in (33) gives

$$\sum_{n=0}^{\infty} \frac{1}{p} P_n\left(\frac{1}{p}\right)(-t)^n = \frac{p(1-t)^2 + \lambda t(1-p) + t(1-t)}{(1-t)(p^2 + (2+\lambda)tp(1-p) + t^2(1-p)^2)}. \tag{40}$$

The right-hand side as a function of p has two poles which are, generally, simple.

$$p^{\pm} = \frac{t}{2}\left(\frac{2t-\lambda-2 \pm \sqrt{4\lambda+\lambda^2}}{(1-t)^2 - t\lambda}\right). \tag{41}$$

Multiplying by the factor e^{px} and inverting the Laplace transform by integrating along the usual complex contour gives the result. By an examination of the poles of the denominator as a function of t, we see that the series (40) converges for all $|t|<|a||p/(p-1)|$ and thus the complex integration along any p-contour of the type $c-i\infty$ to $c+i\infty$, $c>1$, is justified. The result is an analytic function of t. Since $|a|<1<|1/a|$ the series (37) converges for $|t|<|a|$.

Note when $\lambda = 0$, (37) yields the standard generating function for the Laguerre polynomials, [3, vol. 2, p. 189, formula (21)]. To see this, observe that

$$(-1)^n \rho_n L_n^{(\lambda)}(x)|_{x=0} = C_n = 1. \tag{42}$$

The Laguerre polynomials are normalized to have the value 1 at $x = 0$. Thus,

$$(-1)^n \rho_n L_n^{(0)}(x) = L_n(x). \tag{43}$$

An interesting inequality for $L_n^{(\lambda)}$ follows from the generating function (37). The singularities of the function on the left are at $t = 1$, $t = -a$, $t = -1/a$. Expressed in terms of θ these are $t = 1$, $t = \mathrm{e}^{-2\theta}$, $t = \mathrm{e}^{2\theta}$. The smallest of these is $\mathrm{e}^{-2\theta}$. Applying Cauchy's inequality shows that for any ε, $0 < \varepsilon < \mathrm{e}^{-2\theta}$

$$|\rho_n L_n^{(\lambda)}| \leqslant \frac{M_\varepsilon}{(\mathrm{e}^{-2\theta} - \varepsilon)^n}, \tag{44}$$

where M_ε is some constant independent of x.

Theorem 5.

$$\mathrm{e}^{-cx} = \sum_{n=0}^{\infty} \alpha_n L_n^{(\lambda)}(x), \tag{45}$$

where

$$\alpha_n = \rho_n (1-a) \left\{ \left(\frac{c}{c+1}\right)^n \frac{a^{n-1}}{(a^{2n-1}-1)} + \left(\frac{c}{c+1}\right)^{n+1} \frac{a^n}{(a^{2n+1}-1)} \right\}, \tag{46}$$

and

$$a = \frac{-\lambda - 2 + \sqrt{\lambda^2 + 4\lambda}}{2}, \qquad \rho_n = \left(\prod_{j=1}^{n} j!(j-1)! \right)^{-1}. \tag{47}$$

The series converges uniformly on compact subsets of the complex x-plane for all $\lambda > 0$, $c > 0$, as can be seen from the estimate (44).

Proof. Denote the generating on the left of (37) by $g(x,t)$. We have

$$\begin{aligned} \langle \mathrm{e}^{-cx}, g(x,t) \rangle &= \sum_{n=0}^{\infty} \rho_n \langle \mathrm{e}^{-cx}, L_n^{(\lambda)}(x) \rangle (-t)^n \\ &= -\frac{a(ac+a+1)}{(a-1)(ct+ac+a)} + \frac{(ac+a+c)}{(a-1)(cat+c+1)}. \end{aligned} \tag{48}$$

Selecting the coefficient of t^n from the last expression above gives the value of $\langle e^{-cx}, L_n^{(\lambda)}(x)\rangle$. We now need the value of $h_n^{(\lambda)}$. However, this is easy to obtain. We have

$$\begin{aligned}\langle g(x,t), g(x,u)\rangle &= \sum_{n=0}^{\infty} \rho_n^2 \langle L_n^{(\lambda)}(x), L_n^{(\lambda)}(x)\rangle (ut)^n = \sum_{n=0}^{\infty} \rho_n^2 h_n^{(\lambda)} (ut)^n \\ &= -\frac{a}{(a-1)^2(1-ut/a^2)} + \frac{a^2+1}{(a-1)^2(1-ut)} - \frac{a}{(a-1)^2(1-a^2ut)}. \end{aligned} \tag{49}$$

The last expression above is obtained by the evaluation of a Laplace transform integral and partial fractions. Selecting the coefficient of $(ut)^n$ from this expression gives $h_n^{(\lambda)}$,

$$h_n^{(\lambda)} = \frac{-(a^{2n-1}-1)(a^{2n+1}-1)}{a^{2n-1}(a-1)^2\rho_n^2}. \tag{50}$$

We then divide $\langle e^{-cx}, L_n^{(\lambda)}(x)\rangle$ by this expression and do a little algebra to get the coefficients given in formula (46).

Many other expansions of special functions in the polynomials $L_n^{(\lambda)}$ can be obtained by similarly manipulating the generating function (37), including expansions for the confluent hypergeometric functions Φ and Ψ analogous to those expansions in Laguerre polynomials given in [3, vol. 2, p. 215].

Marcellán et al. have shown that the Laguerre–Sobolev polynomials for general α satisfy a third-order (four-term) recurrence relation. One starts with the equation [4] (3.6), which relates the general Sobolev–Laguerre polynomials $Q_n^{(\alpha)}(x)$ (orthogonal with respect to the inner product $(*)$) and the monic Laguerre polynomials $\hat{L}_n^{(\alpha)}(x)$. In the notation of [4],

$$\hat{L}_n^{(\alpha)}(x) + n\hat{L}_{n-1}^{(\alpha)}(x) = Q_n^{(\alpha)}(x) + d_{n-1}(\lambda) Q_{n-1}^{(\alpha)}(x). \tag{51}$$

$d_n(\lambda)$ is a ratio of two Pollaczek polynomials. Now replace n by $n-1$ then by $n-2$ and each time use the recurrence for $\hat{L}_n^{(\alpha)}(x)$ to express $\hat{L}_{n-2}^{(\alpha)}(x)$ in terms of $\hat{L}_n^{(\alpha)}(x)$ and $\hat{L}_{n-1}^{(\alpha)}(x)$. Eliminating the latter two quantities from the set of three equations yields the four-term recursion relation given in [4, p. 258]:

$$\begin{aligned} &Q_n^{(\alpha)}(x) + \left[(2n+\alpha-2-x) + \frac{q_{n-2} n(n+\alpha-1)}{q_{n-1}}\right] Q_{n-1}^{(\alpha)}(x) \\ &\quad + (n-1)(n+\alpha-2)\left[1 + \frac{q_{n-3}(2n+\alpha-2-x)}{q_{n-2}}\right] Q_{n-2}^{(\alpha)}(x) \\ &\quad + \frac{q_{n-4}(n-1)(n-2)}{q_{n-3}}(n+\alpha-2)(n+\alpha-3) Q_{n-3}^{(\alpha)}(x) = 0. \end{aligned} \tag{52}$$

Using the recurrence in Proposition 3.4 of [4], one finds that $q_n(\lambda)$ may be expressed a linear combination of Pollaczek polynomials

$$q_n(\lambda) = (\alpha+1)_n \left[P_n^{(1-\alpha/2)}\left(\frac{\lambda}{2}+1; -\alpha/2, \alpha/2, \alpha\right) - \frac{1}{(\alpha+1)} P_{n-1}^{(1-\alpha/2)}\left(\frac{\lambda}{2}+1; -\alpha/2, \alpha/2, \alpha+1\right)\right], \tag{53}$$

see also [2, p. 185; 3, vol. 2, p. 220]. There are at least two values of λ for which the above expression simplifies. When $\lambda=0$ the explicit value of the Pollaczek polynomials above can be found from the formulas [5, (12); 6, (4.4)]. The former provides a formula

$$ {}_3F_2\left(\begin{matrix} -n, a, \delta \\ a+1, \delta+1 \end{matrix}; 1\right) = \frac{n!a\delta}{(\delta-a)}\left[\frac{1}{(a)_{n+1}} - \frac{1}{(\delta)_{n+1}}\right], \tag{54} $$

that can be used in the explicit formula for the Pollaczek polynomial given in [6, (4.4)]. We find that for the special case $b=-a$,

$$ P_n^{(\lambda)}(1; a, -a, c) = \frac{c}{(2\lambda-1)}\left[\frac{(c+2\lambda-1)_{n+1}}{(c)_{n+1}} - 1\right]. \tag{55} $$

Using this formula in (53) shows

$$ q_n(0) = (\alpha+1)_n, \tag{56} $$

and for this case (52) is easily found to be an iteration of the recurrence of the classical monic Laguerre polynomial.

A simplification of $q_n(\lambda)$ also occurs in the special case $\alpha=0$, since

$$ P_n^{(1)}\left(\frac{\lambda}{2}+1; 0, 0, c\right) = U_n\left(\frac{\lambda}{2}+1\right), \tag{57} $$

where U_n denotes the Chebyshev polynomial of the second kind. As is well known, these polynomials are elementary functions,

$$ U_n\left(\frac{\lambda}{2}+1\right) = (-1)^n \frac{a^{n+1} - a^{-n-1}}{a - a^{-1}}, \quad a = \frac{-\lambda - 2 + \sqrt{\lambda^2 + 4\lambda}}{2}. \tag{58} $$

We find that when $\alpha=0$,

$$ q_n(\lambda) = n!(-1)^n \frac{a^{n+1} - a^{-n}}{a-1}, \quad a = \frac{-\lambda - 2 + \sqrt{\lambda^2 + 4\lambda}}{2}. \tag{59} $$

Eq. (15) shows

$$ a = -e^{-2\theta}, \tag{60} $$

so

$$ q_n(\lambda) = n! A_{n+1}, \tag{61} $$

$$ A_n = \frac{\cosh(2n-1)\theta}{\cosh\theta} = \frac{(\sqrt{1+\lambda/4} + \sqrt{\lambda}/2)^{2n-1} + (\sqrt{1+\lambda/4} - \sqrt{\lambda}/2)^{2n-1}}{2\sqrt{1+\lambda/4}}. $$

The resulting four-term recurrence relation for the monic ordinary Laguerre–Sobolev polynomials is quite simple:

$$\begin{aligned}
&Q_n^{(0)}(x)+\left[(2n-2-x)+\frac{nA_{n-1}}{A_n}\right]Q_{n-1}^{(0)}(x)\\
&+(n-1)\left[(n-2)+\frac{(2n-2-x)A_{n-2}}{A_{n-1}}\right]Q_{n-2}^{(0)}(x)\\
&+\frac{(n-1)(n-2)^2A_{n-3}}{A_{n-2}}Q_{n-3}^{(0)}(x)=0.
\end{aligned}\tag{62}$$

A referee has pointed out that this recurrence can be obtained directly from the generating function (37) without recourse to Pollaczek polynomials. Rewriting that equation, we have

$$\begin{aligned}
&(1-t)\sum_{n=0}^{\infty}\rho_n L_n^{(\lambda)}(x)(-t)^n\\
&=\frac{a e^{xt/(t+a)}}{(a-1)}-\frac{e^{xt/(t+1/a)}}{(a-1)}\\
&=\frac{a}{a-1}\left(1+\frac{t}{a}\right)\sum_{n=0}^{\infty}L_n(x)\left(-\frac{t}{a}\right)^n-\frac{1}{a-1}(1+at)\sum_{n=0}^{\infty}L_n(x)(-at)^n.
\end{aligned}\tag{63}$$

Comparing coefficients of $(-t)^n$ gives

$$\rho_n L_n^{(\lambda)}(x)+\rho_{n-1}L_{n-1}^{(\lambda)}(x)=\frac{1}{a-1}\left(\frac{1}{a^{n-1}}-a^n\right)(L_n(x)-L_{n-1}(x)).\tag{64}$$

Now,

$$\frac{1}{a-1}\left(\frac{1}{a^{n-1}}-a^n\right)=(-1)^n A_n,\qquad \rho_n L_n^{\lambda}(x)=\frac{A_n}{n!}Q_n^{(0)}(x).\tag{65}$$

When (65) is used and the Laguerre polynomials are eliminated from the relation (64), one recovers the recurrence (62).

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References

[1] J. Brenner, Über eine Erweiterung des Orthogonalitätbegriffes bei Polynomen, Proc. Conf. Constructive Theory of Functions, Akadémiai Kiadó, Budapest, 1972.
[2] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.

[3] A. Erdélyi et al., Higher Transcendental Functions, vols. 1 and 2, McGraw-Hill, New York, 1953.
[4] F. Marcellán, T.E. Pérez, M.A. Piñar, Laguerre–Sobolev orthogonal polynomials, J. Comp. Appl. Math. 71 (1996) 249–265.
[5] J. Wimp, B. Beckermann, Families of some two-point Padé approximants and some ${}_4F_3(1)$ identities, SIAM J. Math. Anal. 26 (1995) 765–773.
[6] J. Wimp, Pollaczek polynomials and Padé approximants: some closed form expressions, J. Comp. Appl. Math. 32 (1990) 301–310.
[7] J. Wimp, Sequence Transformations and Their Applications, Academic Press, New York, 1981.